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Model of a minimal risk portfolio under hybrid uncertainty

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Introduction

We investigate the architecture of some models for a minimal risk portfolio under conditions of hybrid uncertainty of possibilistic-probabilistic type.

In our work attention is paid to the study of situations when the interaction of fuzzy model factors is described by both the strongest and weakest t-norms, which allows us to assess the range of risk changes and the behavior of a set of acceptable portfolios, that is, to manage uncertainty when making investment decisions.

Necessary mathematical apparatus

Definition 1. *Fuzzy random variable* $Y(\omega,\gamma)$ *is a real function* $Y:\Omega\times\Gamma\to E^1$ σ*-measurable for each fixed* γ*, where*

$$\mu_Y(\omega,t)=\pi\{\gamma\in\Gamma:Y(\omega,\gamma)=t\}$$

is called its distribution function.

It follows from Definition 1 that the distribution function of a fuzzy random variable depends on a random parameter, that is, it is a random function.

Definition 2. *Let* $Y(\omega,\gamma)$ *be a fuzzy random variable. Its expected value* $E[Y]$ *is a fuzzy variable with possibility distribution function*

$$\mu_{E[Y]}(t)=\pi\{\gamma\in\Gamma:E[Y(\omega,\gamma)]=t\},$$

where E *is the expectation operator*

$$E[Y(\omega,\gamma)]=\int_\Omega Y(\omega,\gamma)\,P(d\omega).$$

Necessary mathematical apparatus

Definition 3. *A covariance of fuzzy random variables X and Y is defined as:*

$$cov\left(X,Y\right)=\frac{1}{2}\int_0^1\Big(cov\left(X_\omega^-\left(\alpha\right),Y_\omega^-\left(\alpha\right)\right)+cov\left(X_\omega^+\left(\alpha\right),Y_\omega^+\left(\alpha\right)\right)\Big)d\alpha,\quad (1)$$

where $X_\omega^-\left(\alpha\right)Y_\omega^-\left(\alpha\right),\ X_\omega^+\left(\alpha\right)Y_\omega^+\left(\alpha\right)$ *are left and right boundaries of* α*-level sets of fuzzy variables* X_ω *and* Y_ω.

Definition 4. *A variance of a fuzzy random variable Y is*

$$D\left[Y\right]=cov(YY).$$

Definition 5. $Z(\gamma)$ *is called an LR-type fuzzy variable, if its distribution function has the form*

$$\mu_Z\left(t\right)=\begin{cases}L\left(\frac{\underline{m}-t}{\underline{d}}\right),\ for\,t<\underline{m},\\ 1,\ for\,\underline{m}\le t\le\overline{m},\\ R\left(\frac{t-\overline{m}}{\overline{d}}\right),\ for\,t>\overline{m}.\end{cases}$$

where $L(t)$ *and* $R(t)$ *are shape functions.*

Necessary mathematical apparatus

We will use triangular norms (t-norms) to aggregate fuzzy information. These norms generalize "min" operation inherent in operations on fuzzy sets and fuzzy variables (see Nguyen et al. (1997)). The following t-norms are of particular interest:

$$T_M(x,y) = \min\{x,y\} \; and \; T_W(x,y) = \begin{cases} \min\{x,y\}, if \max\{x,y\} = 1, \\ 0, else, \end{cases}$$

T_M is called the strongest, and T_W is called the weakest t-norm, since for any arbitrary t-norm T and $\forall x, y \in [0,1]$, the following inequality holds (see, for example, Nguyen et al. (1997)):

$$T_W(x,y) \leq T(x,y) \leq T_M(x,y).$$

Portfolio return under hybrid uncertainty of possibilistic-probabilistic type

Under conditions of hybrid uncertainty of possibilistic-probabilistic type, the return on an investment portfolio can be represented by a fuzzy random function

$$R_p(w, \omega, \gamma) = \sum_{i=1}^{n} R_i(\omega, \gamma) w_i \,, \qquad (2)$$

which is a linear function of equity shares $w = (w_1, \dots, w_n)$ in the portfolio. Here $R_i(\omega, \gamma)$ are fuzzy random variables that model the returns of individual financial assets with the help of shift-scale representation (see Yazenin (2016)):

$$R_i(\omega, \gamma) = a_i(\omega) + \sigma_i(\omega) Z_i(\gamma). \qquad (3)$$

Expected Portfolio Return Under Conditions of Hybrid Uncertainty

Further we assume that fuzzy variables $Z_i(\gamma) = \left[\underline{m}_i, \overline{m}_i, \underline{d}_i, \overline{d}_i\right]_{LR}$ in representation (3) are mutually T-related, where $T \in \{T_M, T_W\}$, and $a_i(\omega), \sigma_i(\omega)$ are shift and scale coefficients – random variables defined on a probability space (Ω, B, **P**), with $\sigma_i(\omega) \geq 0$. Then possibilities distribution of the portfolio return (2) takes the following form

$$R_p^T(w, \omega, \gamma) = \left[\underline{m}_{R_p}(w, \omega), \overline{m}_{R_p}(w, \omega), \underline{d}_{R_p^T}(w, \omega), \overline{d}_{R_p^T}(w, \omega)\right]_{LR}, \qquad (4)$$

where

$$\underline{m}_{R_p}(w, \omega) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right) w_i\,, \qquad \overline{m}_{R_p}(w, \omega) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right) w_i\,,$$

and the coefficients of fuzziness take the form depending on the type of T:

$$\underline{d}_{R_p^M}(w, \omega) = \sum_{i=1}^{n} \sigma_i(\omega)\underline{d}_i w_i\,, \qquad \overline{d}_{R_p^M}(w, \omega) = \sum_{i=1}^{n} \sigma_i(\omega)\overline{d}_i w_i\,,$$

when $T = T_M$, and in case of $T = T_W$:

$$\underline{d}_{R_p^W}(w, \omega) = \max_{i=1\ldots n}\left\{\sigma_i(\omega)\underline{d}_i w_i\right\}, \qquad \overline{d}_{R_p^W}(w, \omega) = \max_{i=1\ldots n}\left\{\sigma_i(\omega)\overline{d}_i w_i\right\}.$$

Expected Portfolio Return Under Conditions of Hybrid Uncertainty

Theorem 1. *Let $T = T_M$. Then expected portfolio return $\hat{R}_p^M(w,\gamma)$ is characterized by the possibilities distribution function*

$$\hat{R}_p^M(w,\gamma) = \mathbf{E}\left[R_p^M(w,\omega,\gamma)\right] = \left[\underline{m}_{\hat{R}_p}(w), \overline{m}_{\hat{R}_p}(w), \underline{d}_{\hat{R}_p^M}(w), \overline{d}_{\hat{R}_p^M}(w)\right]_{LR},$$

where

$$\underline{m}_{\hat{R}_p}(w) = \sum_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \underline{m}_i) w_i, \qquad \overline{m}_{\hat{R}_p}(w) = \sum_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \overline{m}_i) w_i,$$

$$\underline{d}_{\hat{R}_p^M}(w) = \sum_{i=1}^{n} \hat{\sigma}_i \underline{d}_i w_i, \overline{d}_{\hat{R}_p^M}(w) = \sum_{i=1}^{n} \hat{\sigma}_i \overline{d}_i w_i, \hat{a}_i = \mathbf{E}[a_i(\omega)], \hat{\sigma}_i = \mathbf{E}[\sigma_i(\omega)].$$

Theorem 2. *Let $T = T_W$. Then expected portfolio return $\hat{R}_p^W(w,\gamma)$ is characterized by the possibilities distribution function*

$$\hat{R}_p^W(w,\gamma) = \mathbf{E}\left[R_p^W(w,\omega,\gamma)\right] = \left[\underline{m}_{\hat{R}_p}(w), \overline{m}_{\hat{R}_p}(w), \underline{d}_{\hat{R}_p^W}(w), \overline{d}_{\hat{R}_p^W}(w)\right]_{LR},$$

where

$$\underline{d}_{\hat{R}_p^W}(w) = \mathbf{E}\left[\max_{i=1\ldots n}\{\sigma_i(\omega)\underline{d}_i w_i\}\right], \overline{d}_{\hat{R}_p^W}(w) = \mathbf{E}\left[\max_{i=1\ldots n}\{\sigma_i(\omega)\overline{d}_i w_i\}\right].$$

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

In accordance with the classical Markowitz (1952) approach, we need to construct a portfolio risk function in the minimal risk portfolio model.

$$F_p^{\tau\mathbf{E}}(w) = \begin{cases} \tau\left\{\hat{R}_p^T(w,\gamma)\ \mathcal{R}\ m_d\right\} \geq \alpha, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n, \end{cases}$$

where $\mathbb{E}_+^n = \{x \in \mathbb{E}^n : x \geq 0\}$, $\hat{R}_p^T(w,\gamma)$ – expected return, $\mathcal{R}$ – crisp relation $\{\geq, =\}$; $\alpha \in (0, 1]$, m_d – level of profitableness, acceptable to an investor, $T \in \{T_M, T_W\}$.

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

The following theorems allow us to construct equivalent deterministic analogs of acceptable portfolio models.

Theorem 3. *Let in the constraint model* $F_p^{\tau\mathbf{E}}$ $\tau = '\pi'$, $\mathcal{R}$ = '$\geq$'. *Then the equivalent deterministic model of acceptable portfolios has the form:*

$$F_p^{\pi\mathbf{E}}(w) = \begin{cases} \sum_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \overline{m}_i) w_i + \overline{d}_{\hat{R}_p^T}(w) * R^{-1}(\alpha) \geq m_d, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n. \end{cases}$$

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

Theorem 4. *Let in the model of acceptable portfolios* $F_p^{\tau \boldsymbol{E}}$ τ= ‘ν’, $\mathcal{R}$ = '≥'. *Then the equivalent deterministic model of acceptable portfolios takes the form*:

$$F_p^{\nu \mathbf{E}}(w) = \begin{cases} \sum_{i=1}^{n} (\hat{a}_i + \hat{\sigma}_i \underline{m}_i) w_i - \underline{d}_{\hat{R}_p^T}(w) * L^{-1}(1-\alpha) \geq m_d, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n. \end{cases}$$

From theorems 3, 4 we get

Corollary 1. $F_p^{\nu \mathbf{E}}(w) \subseteq F_p^{\pi \mathbf{E}}(w)$.

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

For further analysis we will need the following notation and concepts. We denote by $t = (t_1, \dots, t_n)$ – a vector whose components are possible values of fuzzy variables $Z_1(\gamma), \dots, Z_n(\gamma)$ respectively. For the strongest t-norm with the possibility of $\mu_{Z_i}(t_i)$, the return of the i-th financial asset is a random variable $Z_i^{t_i}(w) = a_i(\omega) + \sigma_i(\omega)t_i$, and $R_p^t(w, \omega) = \sum_{i=1}^{n}(a_i(\omega) + \sigma_i(\omega)t_i)w_i$ is return on the portfolio with the possibility of $\mu_p(t) = \min_{1 \le i \le n}\{\mu_{Z_i}(t_i)\}$.

Then, following Yazenin (2016), with the possibility of $\mu_p(t)$, the expected return and risk of the portfolio are determined by the formulas

$$m_{R_p}(w, t) = \mathbf{E}\left[R_p^t(w, \omega)\right] = \sum_{i=1}^{n}(\hat{a}_i + \hat{\sigma}_i t_i)w_i$$

and

$$d_{R_p}(w, t) = \mathbf{E}\left[\left(R_p^t(w, \omega) - m_{R_p}(w, t)\right)^2\right],$$

respectively.

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

Using standard transformations we obtain the following formula for the variance with the possibility of $\mu_p(t)$ (see, for example, Yazenin (2016)):

$$d_{R_p}(w, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(t_i, t_j) w_i w_j \,,$$

in which

$$c_{ij}(t_i, t_j) = C_{a_i a_j} + C_{a_j \sigma_i} t_j + C_{\sigma_j a_i} t_i + C_{\sigma_i \sigma_j} t_i t_j,$$

$$C_{a_i a_j} = cov(a_i, a_j), C_{a_j \sigma_i} = cov(a_j, \sigma_i), C_{\sigma_j a_i} = cov(\sigma_j, a_i), C_{\sigma_i \sigma_j} = cov(\sigma_i, \sigma_j).$$

Function $d_{R_p}(w, t)$ has properties that are due to the properties of the covariance matrix **C** with elements $c_{ij}(t_i, t_j)$:

- $d_{R_p}(w, t)$ is a convex function on w for a fixed t;
- for any vectors w and t function $d_{R_p}(w, t)$ is nonnegative;
- $d_{R_p}(w, t)$ is a convex function on t for a fixed w.

The function $d_{R_p}(w, t)$ can be written with the help of matrix **C** as

$$d_{R_p}(w, t) = (\mathbf{C}\, w,\, w).$$

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

Lemma 1. *Let in the constraint model* $F_p^{\tau\boldsymbol{P}}(\mathcal{w})$ *random parameters are normally distributed:* $a_i(\omega) \in \mathcal{N}_p\left(\hat{a}_i, \hat{d}_{a_i}\right)$, $\sigma_i(\omega) \in \mathcal{N}_p\left(\hat{\sigma}_i, \hat{d}_{\sigma_i}\right), i = 1, \dots, n$; *fuzzy parameters* $Z_1(\gamma), \dots, Z_N(\gamma)$ *are* T_M*-related,* $\mu_p(t) \geq \alpha_0$. *Then with the possibility of* $\mu_p(t)$ *system of restrictions* $F_p^{\tau\boldsymbol{P}}(\mathcal{w})$ *is equivalent to the system*

$$F_p^{\mu\mathbf{P}}(\mathcal{w}) = \begin{cases} m_{R_p}(\mathcal{w}, t) + \beta_0 \sqrt{d_{R_p}(\mathcal{w}, t)} \geq m_d, \\ \sum_{i=1}^{n} \mathcal{w}_i = 1, \\ \mathcal{w} \in \mathbb{E}_+^n, \end{cases}$$

where β_0 *– is a solution to the equation* $\mathcal{F}_0^1(x) = 1 - p_0$, *and* $\mathcal{F}_0^1(x)$ *– is a function of the standard normal probabilities distribution.*

Models of acceptable portfolios under hybrid uncertainty of possibilistic-probabilistic type

The Lemma proved above allows us to prove a theorem with the help of which an equivalent deterministic analog of the system $F_p^{\tau\mathbf{P}}(w)$ can be constructed.

Theorem 5. *Let in the constraint model* $F_p^{\tau\boldsymbol{P}}(w)$ *random parameters are normally distributed:* $a_i(\omega) \in \mathcal{N}_p(\hat{a}_i, \hat{d}_{a_i})$, $\sigma_i(\omega) \in \mathcal{N}_p(\hat{\sigma}_i, \hat{d}_{\sigma_i}), i = 1, \dots, n$; *fuzzy parameters* $Z_1(\gamma), \dots, Z_N(\gamma)$ *are* T_M *-related,* $\tau = '\pi'$. *Then the system of restrictions* $F_p^{\tau\boldsymbol{P}}(w)$ *is equivalent to the system*

$$F_p^{\pi\mathbf{P}}(w) = \begin{cases} m_{R_p}(w, t^+) + \beta_0\sqrt{d_{R_p}(w, t^+)} \geq m_d, \\ \sum_{i=1}^{n} w_i = 1, \\ w \in \mathbb{E}_+^n, \end{cases}$$

where $m_{R_p}(w, t^+) = \sum_{i=1}^{n}(\hat{a}_i + \hat{\sigma}_i t_i^+)w_i$, $d_{R_p}(w, t^+) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(t_i^+, t_j^+)\, w_i w_j$ *, and* t_i^+, t_j^+ *are right borders of* α_0*-level sets of fuzzy variables* $Z_i(\gamma), Z_j(\gamma)$*, respectively.*

Assessment of portfolio risk with hybrid uncertainty

In accordance with the indicated approach to determining second-order moments, we can determine the variance of the portfolio to assess its risk. We need to obtain formulas (variances) for the case of both the strongest and weakest t-norms.

With $T = T_M$ formula (1) takes the form

$$D_p^M(w) = \frac{1}{2}\int_0^1 \left(\mathbf{D}\left[R_p^{M-}(w,\omega,\alpha)\right] + \mathbf{D}\left[R_p^{M+}(w,\omega,\alpha)\right]\right)d\alpha,$$

where $R_p^{M-}(w,\omega,\alpha)$ and $R_p^{M+}(w,\omega,\alpha)$ – are left and right boundaries of α-level set of fuzzy random variable $R_p^M(w,\omega,\alpha)$:

$$R_p^{M-}(w,\omega,\alpha) = \sum_{i=1}^{n}\left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right)w_i - \sum_{i=1}^{n}\sigma_i(\omega)\underline{d}_i w_i * L^{-1}(\alpha),$$

$$R_p^{M+}(w,\omega,\alpha) = \sum_{i=1}^{n}\left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right)w_i + \sum_{i=1}^{n}\sigma_i(\omega)\overline{d_i} w_i * R^{-1}(\alpha).$$

Assessment of portfolio risk with hybrid uncertainty

If all random parameters of distributions are independent, then after standard transformations we get the formula for the variance:

$$D_p^M(w) = \sum_{i=1}^{n}\Bigg(\mathbf{D}[a_i(\omega)] + \frac{1}{2}\mathbf{D}[\sigma_i(\omega)]\Bigg(\underline{m}_i^2 + \overline{m}_i^2 + 2\overline{m}_i\overline{d}_i\int_0^1 R^{-1}(\alpha)d\alpha - 2\underline{m}_i\underline{d}_i\int_0^1 L^{-1}(\alpha)d\alpha + \overline{d}_i^2\int_0^1\left(R^{-1}(\alpha)\right)^2 d\alpha + \underline{d}_i^2\int_0^1\left(L^{-1}(\alpha)\right)^2 d\alpha\Bigg)\Bigg)w_i^2.$$

Assessment of portfolio risk with hybrid uncertainty

Note that if in fuzzy random variables the fuzzy components are given by LR-type fuzzy numbers with the same left and right shapes and coefficients of fuzziness, i.e. $L(\alpha) = R(\alpha) = S(\alpha), \forall \alpha$ and $\overline{m}_i = \underline{m}_i = m_i, \overline{d}_i = \underline{d}_i = d_i, i = 1, \dots, n$, then the variance formula can be simplified:

$$D_p^M(w) = \sum_{i=1}^{n} \left(\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)] \left(m_i^2 + d_i^2 \int_0^1 \left(S^{-1}(\alpha)\right)^2 d\alpha \right) \right) w_i^2.$$

Example 1. In case when shift and scale coefficients $a_i(\omega), \sigma_i(\omega)$ are uniformly distributed over the segment [0,1] and independent, and the shape function $S(t) = max\{0, 1 - t\}, t \geq 0$, we obtain the following formula for the variance:

$$D_p^M(w) = \frac{1}{12} \sum_{i=1}^{n} \left(m_i^2 + \frac{1}{3} d_i^2 + 1 \right) w_i^2.$$

Assessment of portfolio risk with hybrid uncertainty

We now define the variance for the t-norm T_W. To do this, we will again use formula (1) to find the covariance of two fuzzy random variables. For the weakest t-norm, the formula for finding the variance takes the form:

$$D_p^W(\mathcal{w}) = \frac{1}{2}\int_0^1 \left(\mathbf{D}\left[R_p^{W-}(\mathcal{w}, \omega, \alpha)\right] + \mathbf{D}\left[R_p^{W+}(\mathcal{w}, \omega, \alpha)\right]\right) d\alpha,$$

where $R_p^{W-}(\mathcal{w}, \omega, \alpha)$ and $R_p^{W+}(\mathcal{w}, \omega, \alpha)$ are respectively left and right boundaries of α-level set of portfolio return – fuzzy random variable $R_p^W(\mathcal{w}, \omega, \gamma)$:

$$R_p^{W-}(\mathcal{w}, \omega, \alpha) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\underline{m}_i\right)\mathcal{w}_i - \max_{i=1\ldots n}\left\{\sigma_i(\omega)\underline{d}_i\mathcal{w}_i\right\} * L^{-1}(\alpha),$$

$$R_p^{W+}(\mathcal{w}, \omega, \alpha) = \sum_{i=1}^{n} \left(a_i(\omega) + \sigma_i(\omega)\overline{m}_i\right)\mathcal{w}_i + \max_{i=1\ldots n}\left\{\sigma_i(\omega)\bar{d}_i\mathcal{w}_i\right\} * R^{-1}(\alpha).$$

Assessment of portfolio risk with hybrid uncertainty

After appropriate transformations, the final formula for variance, which allows us to determine the risk of the portfolio, has the form:

$$
\begin{aligned}
&D_p^W(w) \\
&= \frac{1}{2}\sum_{i=1}^{n} w_i^2\left(\mathbf{D}\left[a_i(\omega)+\sigma_i(\omega)\underline{m}_i\right]+\mathbf{D}\left[a_i(\omega)+\sigma_i(\omega)\overline{m}_i\right]\right) \\
&+\frac{1}{2}\mathbf{D}\left[\max_{j=1\dots n}\{\sigma_j(\omega)\overline{d}_j w_j\}\right]\int_0^1\left(R^{-1}(\alpha)\right)^2 d\alpha+\frac{1}{2}\mathbf{D}\left[\max_{j=1\dots n}\{\sigma_j(\omega)\underline{d}_j w_j\}\right]\int_0^1\left(L^{-1}(\alpha)\right)^2 d\alpha \\
&+\sum_{1\le i<j\le n} w_i w_j\left(cov\left(\left(a_i(\omega)+\sigma_i(\omega)\underline{m}_i\right),\left(a_j(\omega)+\sigma_j(\omega)\underline{m}_j\right)\right)\right. \\
&\left.+\,cov\left(\left(a_i(\omega)+\sigma_i(\omega)\overline{m}_i\right),\left(a_j(\omega)+\sigma_j(\omega)\overline{m}_j\right)\right)\right) \\
&+\sum_{i=1}^{n} w_i\left(\int_0^1 R^{-1}(\alpha)d\alpha\, cov\left(\left(a_i(\omega)+\sigma_i(\omega)\overline{m}_i\right),\max_{j=1\dots n}\{\sigma_j(\omega)\overline{d}_j w_j\}\right)\right. \\
&\left.-\int_0^1 L^{-1}(\alpha)d\alpha\, cov\left(\left(a_i(\omega)+\sigma_i(\omega)\underline{m}_i\right),\max_{j=1\dots n}\{\sigma_j(\omega)\underline{d}_j w_j\}\right)\right).
\end{aligned}
$$

Assessment of portfolio risk with hybrid uncertainty

If all random parameters of distributions are independent, then:

$$D_p^W(w)$$
$$= \frac{1}{2}\sum_{i=1}^{n} w_i^2\left(2\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)]\left(\underline{m}_i^2 + \overline{m}_i^2\right)\right)$$
$$+ \frac{1}{2}\mathbf{D}\left[\max_{j=1\ldots n}\left\{\sigma_j(\omega)\overline{d}_j w_j\right\}\right]\int_0^1\left(R^{-1}(\alpha)\right)^2 d\alpha + \frac{1}{2}\mathbf{D}\left[\max_{j=1\ldots n}\{\sigma_j(\omega)\underline{d}_j w_j\}\right]\int_0^1\left(L^{-1}(\alpha)\right)^2 d\alpha$$
$$+ \sum_{i=1}^{n} w_i\left(\int_0^1 R^{-1}(\alpha)d\alpha\, cov\left((a_i(\omega) + \sigma_i(\omega)\overline{m}_i), \max_{j=1\ldots n}\left\{\sigma_j(\omega)\overline{d}_j w_j\right\}\right)\right.$$
$$\left. - \int_0^1 L^{-1}(\alpha)d\alpha\, cov\left((a_i(\omega) + \sigma_i(\omega)\underline{m}_i), \max_{j=1\ldots n}\{\sigma_j(\omega)\underline{d}_j w_j\}\right)\right).$$

Assessment of portfolio risk with hybrid uncertainty

If in all distributions the fuzzy component is given by LR-type fuzzy numbers with the same left and right shape functions and coefficients of fuzziness, then the terms with covariance are mutually eliminated and the variance formula can be simplified:

$$D_p^W(w) = \sum_{i=1}^{n} w_i^2 \left(\mathbf{D}[a_i(\omega)] + \mathbf{D}[\sigma_i(\omega)] m_i^2 \right) + \mathbf{D}\left[\max_{j=1\ldots n} \{ \sigma_j(\omega) d_j w_j \} \right] \int_0^1 \left(S^{-1}(\alpha) \right)^2 d\alpha. \quad (5)$$

Assessment of portfolio risk with hybrid uncertainty

Example 2. Let the shift and scale coefficients $a_i(\omega), \sigma_i(\omega)$ are uniformly distributed over the segment [0,1] and independent the formula (5) under the corresponding assumptions and for $S(t) = max\{0, 1 - t\}$, $t \geq 0$ takes the form:

$$D_p^W(w) = \frac{1}{12}\sum_{i=1}^{n} w_i^2(m_i^2 + 1) + \frac{1}{3}\left(EMax2(dw) - \left(EMax(dw)\right)^2\right),$$

where

$$EMax(dw) \coloneqq \mathbf{E}\left[\max_{i=1\dots n}\{\sigma_i(\omega)d_i w_i\}\right] = \sum_{i=1}^{n} \frac{(dw)_{(i)}^{n-i+1}}{(n-i+1)(n-i+2)(dw)_{(i+1)}\cdots(dw)_{(n)}},$$

$$EMax2(dw) \coloneqq \mathbf{E}\left[\left(\max_{i=1\dots n}\{\sigma_i(\omega)d_i w_i\}\right)^2\right] = \sum_{i=1}^{n} \frac{2(dw)_{(i)}^{n-i+2}}{(n-i+2)(n-i+3)(dw)_{(i+1)}\cdots(dw)_{(n)}},$$

and $(\mathrm{d}w)_{(1)}, \dots, (\mathrm{d}w)_{(\mathrm{n})}$ is an ascending permutation of elements $\{\mathrm{d}_1 w_1, \dots, \mathrm{d}_\mathrm{n} w_\mathrm{n}\}$.

Minimum Risk Portfolio Models

Based on the results presented above, the minimum risk portfolio models can be written as:

$$D_p^T(w) \rightarrow min, \tag{6}$$

$$w \in F_p(w), \tag{7}$$

where $F_p(w) \in \{F_p^{\mu\mathbf{E}}, F_p^{\nu\mathbf{E}}, F_p^{\mu\mathbf{P}}, F_p^{\nu\mathbf{P}}\}$.

Numerical Calculations

We consider an example of two-dimensional portfolio ($n = 2$) . Let $Z_1 = [2.2, 2.2, 0.3, 0.3]_{LR}$, $Z_2 = [1.2, 1.2, 0.4, 0.4]_{LR}$, $L(t) = R(t) = \max\{0, 1 - t\}, t \geq 0, \alpha = 0.75$. Recall that all $a_i(\omega), \sigma_i(\omega)$ are independent random variables with a uniform distribution on the segment [0,1]. We first specify the minimum risk portfolio models for the weakest t-norm:

$$\frac{73}{150} w_1^2 + \frac{61}{300} w_2^2 + \frac{1}{3}\left(EMax2(dw) - \left(EMax(dw)\right)^2\right) \to min,$$

$$F_p^{\pi\mathbf{E}}(w) = \begin{cases} 1.6w_1 + 1.1w_2 + 0.25 * EMax(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0, \end{cases}$$

and in the context of a necessity measure:

$$\frac{73}{150} w_1^2 + \frac{61}{300} w_2^2 + \frac{1}{3}\left(EMax2(dw) - \left(EMax(dw)\right)^2\right) \to min,$$

$$F_p^{\nu\mathbf{E}}(w) = \begin{cases} 1.6w_1 + 1.1w_2 - 0.75 * EMax(dw) \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0. \end{cases}$$

Numerical Calculations

Let us now consider the same problem for the strongest t-norm. Under the assumptions made, the equivalent deterministic analog of the minimum risk portfolio (6)-(7) in the context of the possibility measure takes the form:

$$\frac{587}{1200}w_1^2 + \frac{187}{900}w_2^2 \to min\,,$$

$$F_p^{\pi\mathbf{E}}(w) = \begin{cases} 1.6375w_1 + 1.15w_2 \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0, \end{cases}$$

and in the context of a necessity measure:

$$\frac{587}{1200}w_1^2 + \frac{187}{900}w_2^2 \to min,$$

$$F_p^{\nu\mathbf{E}}(w) = \begin{cases} 1.4875w_1 + 0.95w_2 \geq m_d, \\ w_1 + w_2 = 1, \\ w_1, w_2 \geq 0. \end{cases}$$

Numerical Calculations

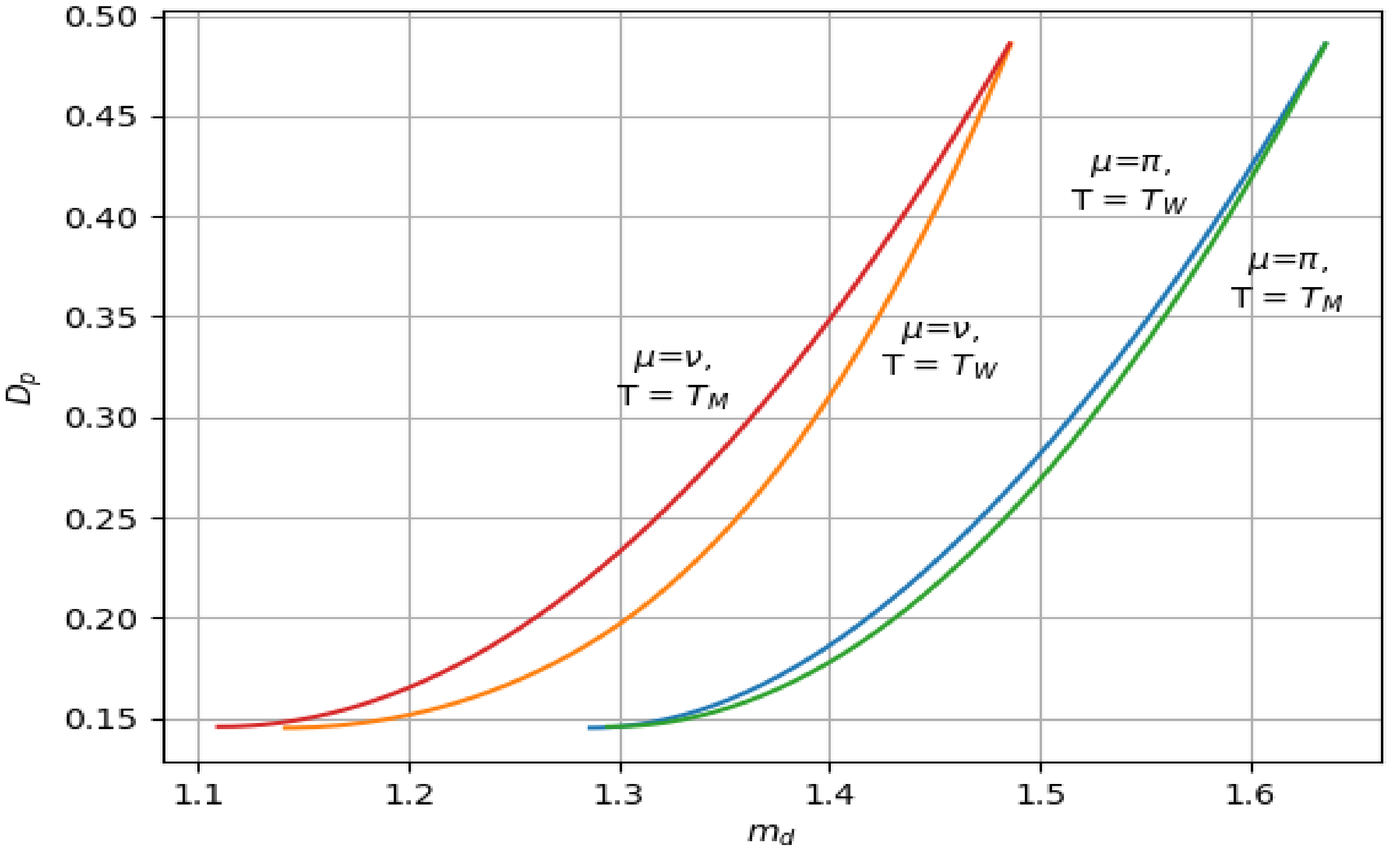

Fig. 1: Sets of quasi-efficient portfolios depending on the measure of possibility/necessity and the t-norm

Numerical Calculations

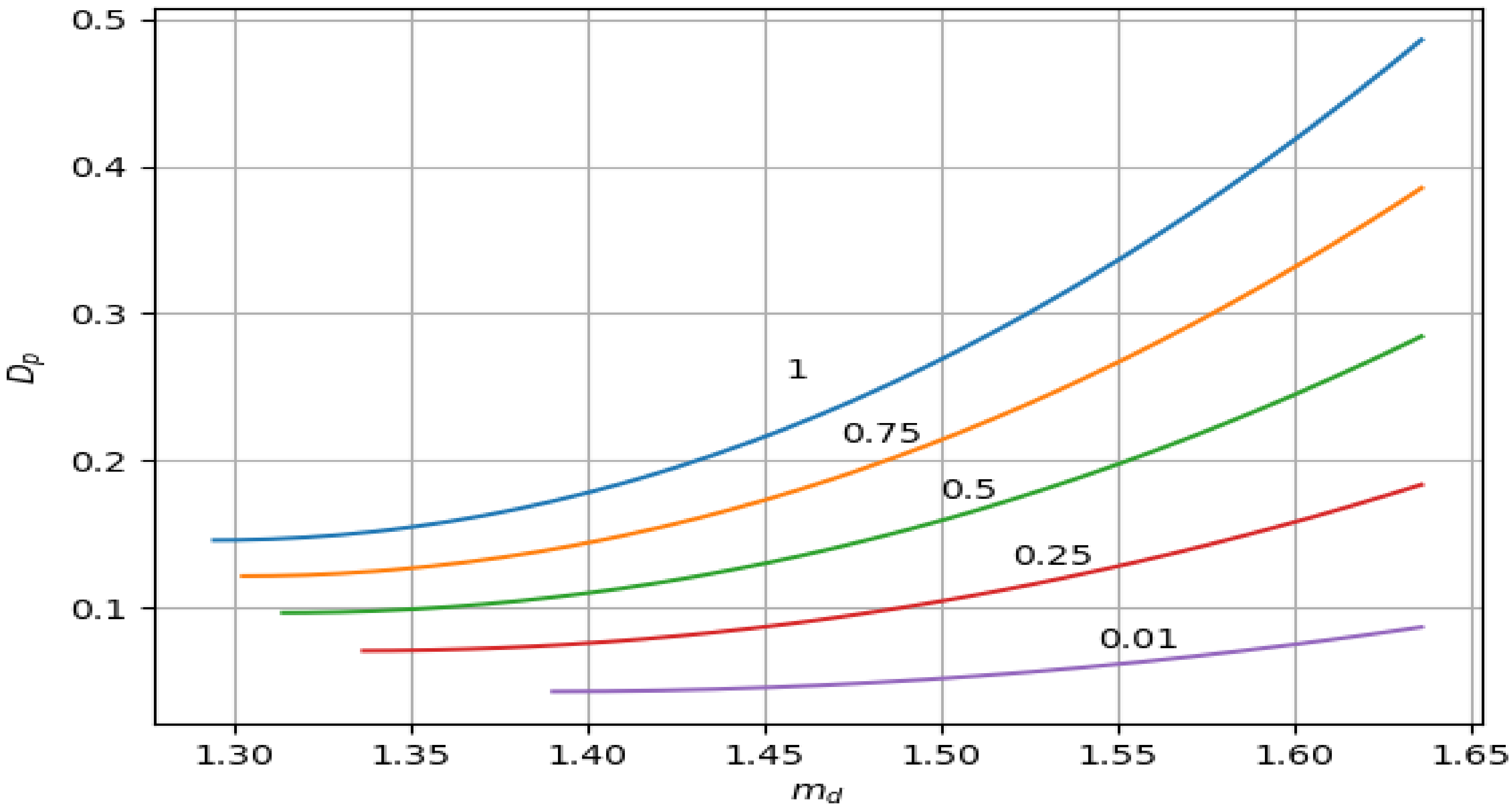

Fig. 2: Influence of the random parameter σ_i (scaling) on the set of quasi-efficient portfolios in the case of the strongest t-norm in the context of the measure of possibility

Thank you!